

AN ALGEBRAIC REDUCTION OF THE ‘SCALING GAP’ IN THE NAVIER-STOKES REGULARITY PROBLEM

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ABSTRACT. It is shown—within a mathematical framework based on the suitably defined scale of sparseness of the super-level sets of the positive and negative parts of the vorticity components, and in the context of a blow-up-type argument—that the ever-resisting ‘scaling gap’ in the 3D Navier-Stokes regularity problem can be reduced by an *algebraic factor*; all preexisting improvements have been logarithmic in nature, regardless of the functional set up utilized. The mathematics presented was inspired by morphology of the regions of intense vorticity/velocity gradients observed in computational simulations of turbulent flows.

1. PROLOGUE

The Navier-Stokes equations (NSE) describing the motion of three-dimensional (3D) incompressible, viscous, Newtonian fluids read

$$(1) \quad \partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p + f,$$

supplemented with the incompressibility condition $\nabla \cdot u = 0$. Here, the vector field u represents the velocity, the scalar field p the pressure, a positive constant ν is the viscosity, and the vector field f an external force.

Henceforth, for simplicity of the exposition, ν will be normalized to 1, f taken to be a potential force, and the spatial domain to be the whole space \mathbb{R}^3 (since this is an irreversible system, the temporal variable lives in the interval $(0, \infty)$). In this setting, the only relevant piece of data is the initial velocity configuration u_0 .

Mathematical theory of the 3D NSE emerged from the groundbreaking work of Leray in the 1930s [Leray1, Leray2, Leray3] in which he established global-in-time existence (no uniqueness) of weak solutions emanating from the initial data with arbitrary large energy, and charted the paths for the study of local-in-time and global-in-time existence and uniqueness of strong solutions within the realms of arbitrary large and appropriately small initial data, respectively.

A central problem in the theory ever since has been the possibility of singularity formation in the system; this could be considered either from the point of view of possible singularity formation in a weak solution or from the point of view of possible finite time blow-up of a local-in-time strong solution, the two being intimately related. This problem is usually referred to as the Navier-Stokes regularity problem.

The Navier-Stokes regularity problem is supercritical, i.e., there is a ‘scaling gap’ (with respect to the unique scaling-invariance of the equations) between any known regularity criterion and the corresponding *a priori* bound. If we focus on $L^\infty((0, T), X)$ -type spaces, perhaps the most illustrative example is in the L^p -scale where the regularity class is $L^\infty((0, T), L^3)$ ([ESS03]; for generalizations see [GKP, Phuc]), and the corresponding *a priori* bound the Leray bound $L^\infty((0, T), L^2)$. Moreover, the scaling gap has been of the fixed ‘size’ in the sense that all the regularity classes are (at best) scaling-invariant while—in contrast—all the known *a priori* bounds are on the scaling level of the Leray bound.

One might try to collect additional clues about possible singularity formation in the 3D NSE from the geometry of turbulent flows. One of the most prominent morphological signatures of this geometry—revealed in high-resolution computational simulations of high (and even moderate; see [SchSree]) Reynolds number-flows—is spatial intermittency of the regions of intense vorticity and velocity gradients.

In order to better understand geometry of the flow mathematically, it is beneficial to study the vorticity-velocity formulation of the 3D NSE,

$$(2) \quad \partial_t \omega + (u \cdot \nabla) \omega = \nu \Delta \omega + (\omega \cdot \nabla) u$$

where the vorticity field ω is given by $\omega = \text{curl } u$. The incompressibility implies that u can be reconstructed from ω by solving $\Delta u = -\text{curl } \omega$, leading to the Biot-Savart law, and closing the system.

Let us start by noting that conceivably a first work in mathematical analysis of the 3D NSE that was motivated by geometry of the flow was a work by Giga and Miyakawa [GiMi] establishing global-in-time existence of small, measure-valued solutions modeling the vortex filaments (the space utilized is a scaling-invariant Morrey-Campanato space).

A conceptually related program aimed at mathematical modeling and analysis of coherent structures in fluid flows via the study of oscillating and localized solutions to the NSE was initiated by Meyer and his students (Brandolese, Cannone, Lemarie-Rieusset, Planchon, ...); this brought in an array of scale-sensitive harmonic analysis tools such as wavelets, atomic decomposition and Littlewood-Paley analysis. A chapter in C.I.M.E. lecture notes by Meyer [Mey] provides an insight into many of the main ideas and results related to this program.

The vortex-stretching term, $(\omega \cdot \nabla)u$, is responsible for a possibly unbounded growth of the vorticity magnitude in 3D; mechanically, the process of vortex stretching in conjunction with incompressibility of the fluid and conservation of the angular momentum amplifies the vorticity, formally, if we erase the vortex-stretching term in the equations, and the only nonlinear term left is the advection term $(u \cdot \nabla)\omega$, an *a priori* bound on the L^2 -norm of the vorticity—which suffices to bootstrap to infinite smoothness—follows in one line ($(\omega \cdot \nabla)u$ is identically zero in 2D).

However, some caution is advised in labeling the vortex-stretching term as a clear-cut ‘bad guy’. While the process of vortex-stretching does amplify the vorticity, it also produces locally anisotropic small scales in the flow (diameter of a vortex tube, thickness of a vortex sheet, *etc.*), and this gives a chance for some form of a locally anisotropic diffusion to engage just on time to prevent the further amplification of the vorticity magnitude and possible singularity formation. As a matter of fact, Taylor [Tay37] concluded his paper from the 1930s with the following thought on turbulent dissipation:

“It seems that the stretching of vortex filaments must be regarded as the principal mechanical cause of the high rate of dissipation which is associated with turbulent motion.”

The stretching is also chiefly responsible for the phenomenon of local coherence of the vorticity direction prominent in coherent vortex structures (e.g., in vortex tubes) which can be viewed as a local manifestation of the quasi low-dimensionality of the regions of intense fluid activity. This observation led to the rigorous mathematical study of locally anisotropic dissipation in the 3D NSE originating in the pioneering work by Constantin [Co94] where he presented a singular integral representation of the stretching factor in the evolution of the vorticity magnitude, featuring a geometric kernel depleted precisely by local coherence of the vorticity direction. This has been referred to as ‘geometric depletion of the nonlinearity’, and was a key in the proof of a fundamental result by Constantin and Fefferman [CoFe93], stating that as long as the vorticity direction (in the regions of intense vorticity) is Lipschitz-coherent, no finite time blow-up can occur.

The Lipschitz-coherence was later scaled down to $\frac{1}{2}$ -Hölder [daVeigaBe02], followed up by a complete spatiotemporal localization to an arbitrary small parabolic cylinder $B(x_0, 2R) \times (t_0 - (2R)^2, t_0)$ [Gr09], and the construction of scaling-invariant, local, hybrid geometric-analytic regularity classes in which the coherence of the vorticity direction is balanced against the growth of the vorticity magnitude [GrGu10-1]. In particular, denoting the vorticity direction by ξ , the following scaling-invariant regularity class is included,

$$\int_{t_0 - (2R)^2}^{t_0} \int_{B(x_0, 2R)} |\omega(x, t)|^2 \rho_{\frac{1}{2}, 2R}^2(x, t) dx dt < \infty$$

where

$$\rho_{\gamma, r}(x, t) = \sup_{y \in B(x, r), y \neq x} \frac{|\sin \angle(\xi(x, t), \xi(y, t))|}{|x - y|^\gamma};$$

this is to be contrasted to an *a priori* bound obtained in [Co90],

$$\int_0^T \int_{\mathbb{R}^3} |\omega(x, t)| |\nabla \xi(x, t)|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_0(x)|^2 dx,$$

which exhibits the same scaling as the Leray class $L^\infty((0, T), L^2)$; another manifestation of the scaling gap.

Let us go back to the spatial intermittency of the regions of intense vorticity, more specifically, to ‘sparseness’ of the vorticity super-level sets. This type of morphology motivated a recent work [Gr13] where it was shown that as long as the vorticity super-level sets, cut at a fraction of the L^∞ -norm, exhibit a suitably defined property of sparseness at the scale comparable to the radius of spatial analyticity measured in L^∞ , no finite time blow-up can occur. (The proof was presented for the velocity formulation, and the argument for the vorticity formulation is completely analogous.)

The concept of sparseness needed in the proof is essentially local existence of a sparse direction (the direction is allowed to depend on the point) at the scale of interest, and it will be referred to as ‘1D sparseness’. The precise definitions of 1D sparseness and a related concept of ‘3D sparseness’ can be found at the beginning of section 3, as well as a remark on their relationship. In the rest of this section, sparseness will refer to 3D sparseness; we copy the definition below for convenience of the reader (m^3 denotes the 3-dimensional Lebesgue measure).

Let S be an open subset of \mathbb{R}^3 , $\delta \in (0, 1)$, x_0 a point in \mathbb{R}^3 , and $r \in (0, \infty)$. S is *3D δ -sparse around x_0 at scale r* if

$$\frac{m^3(S \cap B(x_0, r))}{m^3(B(x_0, r))} \leq \delta.$$

Since we are interested in the super-level sets of a vector field cut at a fraction of the L^∞ -norm, at the scale comparable to some inverse power of the L^∞ -norm, there are several parameters floating around: the fraction at which we cut (call it λ), the ratio of sparseness δ and a constant multiplying the inverse power of the L^∞ -norm (call it $\frac{1}{c}$). More precisely, the class of functions of interest, X_α , is defined as follows.

Definition 1. For a positive exponent α , and a selection of parameters λ in $(0, 1)$, δ in $(0, 1)$ and $c_0 > 1$, the class of functions $X_\alpha(\lambda, \delta; c_0)$ consists of bounded, continuous functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ subjected to the requirement that the super-level set

$$(3) \quad \left\{ x \in \mathbb{R}^3 : |f(x)| > \lambda \|f\|_\infty \right\}$$

is δ -sparse around any point x_0 at scale $\frac{1}{c} \frac{1}{\|f\|_\infty^\alpha}$, for some c , $\frac{1}{c_0} \leq c \leq c_0$.

It is instructive to make a brief comparison of the scale of classes X_α with the scale of Lorentz $L^{p, \infty}$ spaces defined by the requirement that the distribution function of f , the Lebesgue measure of the super-level set cut at level β , decays at least as the inverse p -power of β . There are two obvious differences:

(i) the definition of X_α involves a specific type of the super-level sets, cut at a fraction of the L^∞ -norm, while the definition of $L^{p,\infty}$ involves all levels β ; however, if one is interested in the behavior of f near a possible singularity, only the large β -levels count,

(ii) of course, the key difference is that the $L^{p,\infty}$ spaces are indifferent to any geometric properties of the super-level sets, and measure only the rate of decay of the volume, while the classes X_α can detect the scale of sparseness of the super-level sets of the highest interest.

Remark 2. It is worth noting that the same phenomenon will occur in all rearrangement-invariant (with respect to the Lebesgue measure) function classes, i.e., the classes that do not differentiate between the functions having identical distribution functions. Typical examples are Lebesgue classes L^p , general Lorentz classes $L^{p,q}$ and Orlicz classes L^ϕ . Various *BMO*-type classes are of course more sensitive to spatial complexity; however, one can still geometrically ‘rearrange’ the spatial structure by any uniformly bounded bi-Lipschitz family of (Lebesgue) measure-preserving homeomorphisms without significantly affecting the *BMO*-type norm in view.

Taking a closer look, on one hand, it is plain that

$$(4) \quad f \in L^{p,\infty} \implies f \in X_\alpha \text{ for } \alpha = \frac{p}{3}$$

(for a given selection of λ and δ , the tolerance parameter c_0 will depend on the $L^{p,\infty}$ -norm of f), on the other hand, in the geometrically worst case scenario (from the point of view of sparseness), the whole super-level set being clumped into a single ball, being in X_α is consistent with being in $L^{3\alpha,\infty}$.

This provides a simple ‘conversion rule’ between the two scales, $\alpha = \frac{p}{3}$, determining the scaling signature of the classes X_α .

Let us decode the scaling distance between the regularity criterion established in [Gr13] (in the vorticity formulation), and any corresponding *a priori* bounds.

Recall that the scale of sparseness required in the criterion is comparable to the lower bound on the radius of spatial analyticity evaluated at a suitable time t prior to (possible) blow-up, namely, $\frac{1}{c} \frac{1}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$, i.e., one needs ω in $X_{\frac{1}{2}}$.

In contrast, the best *a priori* bound available is ω in $X_{\frac{1}{3}}$; this follows simply from the *a priori* bound on the L^1 -norm of the vorticity ([Co90]) and Chebyshev’s inequality (an $L \log L$ -bound on the vorticity is available under a very weak assumption that the vorticity direction is in a local, logarithmically-weighted *BMO* space that allows for discontinuities [BrGr13-2]).

Applying the conversion rule, $X_{\frac{1}{2}}$ corresponds to $L^{\frac{3}{2},\infty}$ and $X_{\frac{1}{3}}$ to $L^{1,\infty}$; since these are the exact ‘boundary points’ within the $L^{p,\infty}$ -scale in the vorticity formulation, we arrive at yet another manifestation of the scaling gap (also, not terribly surprising as the derivation of the *a priori* bound, ω in $X_{\frac{1}{3}}$, uses no geometric tools).

Back to the drawing board. Recall that computational simulations of turbulent flows indicate a high degree of local anisotropy present in the vorticity field, manifested through the formation of various coherent vortex structures (e.g., vortex sheets and vortex tubes).

Regularity class	A priori bound	Energy class
$u \in L^\infty(0, T; L^3)$	$u \in L^\infty(0, T; L^2)$	$u \in L^\infty(0, T; L^2)$
$\omega \in L^\infty(0, T; L^{\frac{3}{2}})$	$\omega \in L^\infty(0, T; L^1)$	$\omega \in L^\infty(0, T; L^1)$
$\omega(\tau) \in X_{\frac{1}{2}}$ for a particular $\tau < T$	$\omega(\tau) \in X_{\frac{1}{3}}$ a.e. $\tau < T$	$\omega(\tau) \in X_{\frac{1}{3}}$ a.e. $\tau < T$
$\omega(\tau) \in Z_{\frac{1}{2}}$ for a particular $\tau < T$	$\omega(\tau) \in Z_{\frac{2}{5}}$ a.e. $\tau < T$ whenever $\ \omega(\tau)\ _\infty$ is large enough	$\omega(\tau) \in Z_{\frac{1}{3}}$ a.e. $\tau < T$

TABLE 1. The scaling distance between regularity and energy classes is consistent across the table. Generally, a priori bounded quantities are in the energy class. Remarkably, Theorem 21 shows the a priori bound in the Z_α class is *algebraically better* than the energy class.

A simple but key observation is that a vector field exhibiting a high degree of local anisotropy is more likely to allow for a considerable discrepancy in sparseness between the full vectorial super-level sets and the super-level sets of the components. As a matter of fact, it is easy to construct locally anisotropic smooth vector fields for which the former are not sparse at any scale, while the latter can feature any predetermined scale of sparseness.

This is our primary motivation for replacing the scale of classes X_α by the scale of classes Z_α based on sparseness of the super-level sets of the positive and negative parts of the vectorial components f_i, f_i^\pm , for $i = 1, 2, 3$.

Definition 3. For a positive exponent α , and a selection of parameters λ in $(0, 1)$, δ in $(0, 1)$ and $c_0 > 1$, the class of functions $Z_\alpha(\lambda, \delta; c_0)$ consists of bounded, continuous functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ subjected to the requirement that all six super-level sets

$$\left\{ x \in \mathbb{R}^3 : f_i^\pm(x) > \lambda \|f\|_\infty \right\}$$

are δ -sparse around any point x_0 at scale $\frac{1}{c} \frac{1}{\|f\|_\infty^\alpha}$, for some $c, \frac{1}{c_0} \leq c \leq c_0$.

(The conversion rule remains the same, $\alpha = \frac{p}{3}$.)

Going back to the vorticity, since this is potentially a much weaker condition, the two natural questions to examine are as follows.

(a) is it possible to obtain an analogous regularity criterion at the same scale of sparseness (compared to the full vectorial super-level sets), or at all?

(b) is it possible to establish an *a priori* bound at a smaller scale of sparseness (compared to the full vectorial case), in the sense of modifying the scaling exponent α by an algebraic factor?

Affirmative answers to (a) and (b) are presented in sections 3 (Theorem 15) and 4 (Theorem 21), respectively, yielding a regularity criterion in $Z_{\frac{1}{2}}$ and an *a priori* bound in $Z_{\frac{2}{5}}$ (with the identical selection of the parameters λ , δ and c_0); since

$$\frac{1}{3} < \frac{2}{5} < \frac{1}{2},$$

this represents an algebraic reduction of the scaling gap in the Z_α -framework!

The various classes and respective bounds are listed in Table 1.

Remark 4. For a quick comparison, note that in the geometrically worst case scenario (for sparseness), being in $Z_{\frac{2}{5}}$ is consistent with being in $L^{\frac{6}{5}, \infty}$.

Remark 5. One should note that this is not the first time that the exponent $\frac{2}{5}$ appeared in the study of formation of ‘small scales’ in turbulent flows. In the context of the 3D inviscid (incompressible) dynamics, the Beale-Kato-Majda (BKM)-type criteria reveal that the formation of infinite spatial gradients/infinitesimally-small spatial scales is necessary for a finite time blow-up. In particular (see a discussion in [Co94]), if we consider the case of an (approximately) self-similar blow-up for the 3D Euler in the form

$$\omega(x, t) \approx \frac{1}{T-t} \Omega\left(\frac{x}{L(t)}\right),$$

then the relevant BKM-type necessary condition for the blow-up reads

$$\int_0^T (L(t))^{-\frac{5}{2}} dt = \infty,$$

and if we focus on the algebraic-type dependence,

$$L(t) = L(0) \left(1 - \frac{t}{T}\right)^p,$$

then the blow-up can be sustained only when $p \geq \frac{2}{5}$. Furthermore, since by dimensional analysis $\omega \sim \frac{1}{T-t}$, this scaling is fully consistent with the vorticity being in $Z_{\frac{2}{5}}$ near a possible singular time T , strongly hinting at the inviscid nature of $Z_{\frac{2}{5}}$.

Remark 6. It is not inconceivable that—at least partially—the discrepancy between the exponents $\frac{2}{5}$ and $\frac{1}{3}$ can be viewed as an indirect measure of the degree of local anisotropy of the vorticity field in the regions of intense fluid activity. It is worth noting that the same line of arguments, applied to the velocity itself, does not yield any reduction of the scaling gap; this is consistent with the picture painted by computational simulations (and fluid mechanics) in which the regions of high velocity magnitude tend to be more isotropic (recall that in a suitable statistical sense, the K41 turbulence phenomenology—based on the velocity—is isotropic and homogeneous, while one way to rationalize the phenomenon of intermittency in turbulence is precisely via the coherent vortex structures).

2. LOCAL-IN-TIME SPATIAL ANALYTICITY OF THE VORTICITY IN L^∞

One of the most compelling manifestations of diffusion in the 3D NSE is the instantaneous spatially-analytic smoothing of the rough initial data. An explicit, sharp lower bound on the radius of (spatial) analyticity of the solution emanating from an initial datum in L^p , for $3 < p < \infty$, was given in [GrKu98]; the method—inspired by the so-called Gevrey class-method introduced by Foias and Temam in [FT]—was based on complexifying the equations and tracking the boundary of the (locally-in-time expanding) domain of analyticity via solving a real system of PDEs satisfied by the real and the imaginary parts of the complexified solutions.

Since the Riesz transforms are not bounded on L^∞ , to obtain an estimate in the case $p = \infty$ without a logarithmic correction requires a different argument (on the real level), see, e.g., [GIM, Ku2] and [Gu] in the real and the complex setting, respectively, in the realm of the velocity-pressure formulation of the 3D NSE.

The L^∞ -argument within the vorticity-velocity description requires a modification, and we present a sketch here, including an estimate on the vortex-stretching term, for completeness. What follows is a modification of the exposition given in [Ku2]. In addition to the initial vorticity ω_0 being bounded, a suitable decay of ω_0 at infinity will be required (we chose ω_0 in L^2 for convenience); however, it is worth noting that this is a ‘soft assumption’, i.e., there will be no quantitative dependence on $\|\omega_0\|_2$ in the proof.

Theorem 7 (real setting). *Let the initial datum ω_0 be in $L^2 \cap L^\infty$. Then there exists a unique mild solution ω in $C([0, T], L^\infty)$ where $T \geq \frac{1}{c} \frac{1}{\|\omega_0\|_\infty}$ for an absolute constant $c > 0$.*

Remark 8. The above theorem is not a ‘pure L^∞ ’-result as we require some global integrability of the initial data; however, it produces an optimal estimate on the time-interval of existence and—in turn—on the radius of spatial analyticity (in the complexified version) which is its primary purpose. As a matter of fact, since the main goal of the paper is to demonstrate an algebraic reduction of the scaling gap in the regularity problem, even a long-established result featuring a logarithmic correction would suffice (see, e.g., [Ku1]).

Sketch of the proof. The vorticity-velocity formulation of the 3D NSE in the components is as follows,

$$(5) \quad \partial_t \omega_j - \Delta \omega_j + u_i \partial_i \omega_j = \omega_i \partial_i u_j, \quad j = 1, 2, 3$$

where u can be recovered from ω by the Biot-Savart law,

$$u_j(x, t) = c \int \epsilon_{j,k,l} \partial_{y_k} \frac{1}{|x - y|} \omega_l(y, t) dy$$

($\epsilon_{j,k,l}$ is the Levi-Civita symbol), making this a closed system for ω . Differentiation yields

$$(6) \quad \partial_i u_j(x, t) = c P.V. \int \epsilon_{j,k,l} \frac{\partial^2}{\partial_{x_i} \partial_{y_k}} \frac{1}{|x - y|} \omega_l(y, t) dy.$$

A key observation is that the kernel

$$\frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x - y|}$$

is a classical Calderon-Zygmund kernel; hence,

$$(7) \quad \|\nabla u(\cdot, t)\|_{BMO} \leq c \|\omega(\cdot, t)\|_{BMO}.$$

Since we are interested in the mild solutions, we rewrite the equations (5) via the action of the heat semigroup,

$$(8) \quad \begin{aligned} \omega_j(x, t) = & \int G(x - y, t) (\omega_0)_j(y) dy \\ & - \int_0^t \int G(x - y, t - s) u_i \partial_i \omega_j(y, s) dy ds \\ & + \int_0^t \int G(x - y, t - s) \omega_i \partial_j u_j(y, s) dy ds, \end{aligned}$$

where G is the heat kernel.

Similarly as in the velocity-pressure description (see, e.g., [GrKu98, Ku1]), the estimates are performed on the sequence of smooth (entire in the spatial variable) global-in-time approximations.

$$(9) \quad \begin{aligned} \omega_j^{(n)}(x, t) = & \int G(x - y, t) (\omega_0)_j(y) dy \\ & - \int_0^t \int G(x - y, t - s) u_i^{(n-1)} \partial_i \omega_j^{(n-1)}(y, s) dy ds \\ & + \int_0^t \int G(x - y, t - s) \omega_i^{(n-1)} \partial_j u_j^{(n-1)}(y, s) dy ds, \end{aligned}$$

supplemented with

$$u_j^{(n)}(x, t) = c \int \epsilon_{j,k,l} \partial_{y_k} \frac{1}{|x - y|} \omega_l^{(n)}(y, t) dy.$$

The goal is to keep $\omega^{(n)}$ in L^∞ and $\nabla u^{(n)}$ in BMO (uniformly in n , locally-in-time), analogously to keeping $u^{(n)}$ in L^∞ and $p^{(n)}$ in BMO in the velocity-pressure formulation.

We present an estimate on the vortex-stretching term $\omega_i^{(n-1)} \partial_j u_j^{(n-1)}$, the term responsible for a potentially unrestricted growth of the vorticity magnitude, in some detail.

An estimate on the vortex-stretching term.

The main ingredients are: a pointwise estimate on the heat kernel G , $G(x, t) \leq c \frac{\sqrt{t}}{(|x| + \sqrt{t})^4}$ (see, e.g., [LR]), a property of a scalar-valued BMO function f featuring a suitable decay at infinity (e.g., being in the closure of the test functions in the uniformly-local L^p for some p , $1 \leq p < \infty$) $\int \frac{|f(x)|}{(|x|+1)^4} dx \leq c \|f\|_{BMO}$ (in general, one has to subtract a local average, for example, over a unit ball B , in which case the inequality takes the form $\int \frac{|f(x) - \frac{1}{|B|} \int_B f|}{(|x|+1)^4} dx \leq c \|f\|_{BMO}$ [St]), and the Calderon-Zygmund relation (6).

Combining the above ingredients implies the following string of inequalities.

$$\begin{aligned}
(10) \quad & \left| \int_0^t \int G(x-y, t-s) \omega_i^{(n)}(y, s) \partial_i u_j^{(n)}(y, s) dy ds \right| \\
& \leq \sup_{s \in (0, t)} \|\omega^{(n)}(s)\|_\infty \int_0^t \int \frac{\sqrt{t-s}}{(|x-y| + \sqrt{t-s})^4} |\partial_i u_j^{(n)}| dy ds \\
& \leq \sup_{s \in (0, t)} \|\omega^{(n)}(s)\|_\infty \int_0^t \int \frac{1}{(|z|+1)^4} |\partial_i u_j^{(n)}| dz ds \\
& \leq c t \sup_{s \in (0, t)} \|\omega^{(n)}(s)\|_\infty \sup_{s \in (0, t)} \|\partial_i u_j^{(n)}(s)\|_{BMO} \\
& \leq c t \sup_{s \in (0, t)} \|\omega^{(n)}(s)\|_\infty \sup_{s \in (0, t)} \|\omega^{(n)}(s)\|_{BMO} \\
& \leq c t \left(\sup_{s \in (0, t)} \|\omega^{(n)}(s)\|_\infty \right)^2.
\end{aligned}$$

(Since $\omega^{(n)}(s)$ is in L^2 , $\partial_i u_j^{(n)}(s)$ is in L^2 as well (by Calderon-Zygmund), and the first BMO -bound is available.)

The estimate on the advection term can be absorbed in the above bound, and we arrive at

$$\sup_{s \in (0, t)} \|\omega^{(n+1)}(s)\|_\infty \leq c \|\omega_0\|_\infty + c t \left(\sup_{s \in (0, t)} \|\omega^{(n)}(s)\|_\infty \right)^2,$$

which yields boundedness of the sequence on the time-interval of the desired length. A similar argument implies the contraction property in $C([0, T], L^\infty)$, concluding the proof.

The complexification argument is analogous to the complexification of the velocity-pressure formulation; for details, see [GrKu98].

The main idea is as follows. Recall that the functions in the sequence of approximations $(\omega^{(n)}, u^{(n)})$ are entire functions in the spatial variable for any $t > 0$. Then, denoting a point in \mathbb{C}^3 by $x + iy$, introduce a related sequence $(\Omega^{(n)}, U^{(n)})$ by

$$\Omega^{(n)}(x, t) = \omega^{(n)}(x + i\alpha t, t) \quad \text{and} \quad U^{(n)}(x, t) = u^{(n)}(x + i\alpha t, t)$$

where $\alpha > 0$ is an optimization parameter.

The real and the imaginary parts of each of the elements in the sequence $(\Omega^{(n)}, U^{(n)})$ satisfy a real system in which the principal linear parts are decoupled, and the only new type of terms generated by the complexification procedure are two lower-order (first order) linear terms coming from the chain rule and the Cauchy-Riemann equations in \mathbb{C}^3 .

Local-in-time estimates on the system are then of the same type as the estimates on the real 3D NSE, resulting in expansion of the domain of analyticity—after the optimization in the parameter α —comparable to \sqrt{t} (the only global-in-time property needed is for the real and imaginary parts of each $\Omega^{(n)}$, on each real-space ‘slice’, to be in L^2 ; this is, as in the real case, guaranteed by the assumption that ω_0 in L^2).

More precisely, we arrive at the following result.

Theorem 9 (complex setting). *Let the initial datum ω_0 be in $L^2 \cap L^\infty$, and M a constant larger than 1. Then there is a constant $c(M) > 1$ such that there exists a unique mild solution ω in $C([0, T], L^\infty)$ where $T \geq \frac{1}{c(M)} \frac{1}{\|\omega_0\|_\infty}$, and for any t in $(0, T]$ the solution ω is the \mathbb{R}^3 -restriction of a holomorphic function ω defined in the domain*

$$\Omega_t = \left\{ x + iy \in \mathbb{C}^3 : |y| < \frac{1}{\sqrt{c(M)}} \sqrt{t} \right\};$$

moreover, $\|\omega(t)\|_{L^\infty(\Omega_t)} \leq M \|\omega_0\|_\infty$.

3. SPATIAL COMPLEXITY OF THE VORTICITY COMPONENTS: A REGULARITY CRITERION

Henceforth, m^n will denote the n -dimensional Lebesgue measure. Let us start with recalling the concepts of 1D and 3D sparseness suitable for our purposes ([Gr13, FGL]).

Let S be an open subset of \mathbb{R}^3 , $\delta \in (0, 1)$, x_0 a point in \mathbb{R}^3 , and $r \in (0, \infty)$.

Definition 10. S is 1D δ -sparse around x_0 at scale r if there exists a unit vector \mathbf{d} in S^2 such that

$$\frac{m^1(S \cap (x_0 - r\mathbf{d}, x_0 + r\mathbf{d}))}{2r} \leq \delta.$$

Definition 11. S is 3D δ -sparse around x_0 at scale r if

$$\frac{m^3(S \cap B(x_0, r))}{m^3(B(x_0, r))} \leq \delta.$$

Remark 12. On one hand, it is straightforward to check that, for any S , 3D δ -sparseness at scale r implies 1D $(\delta)^{\frac{1}{3}}$ -sparseness at scale ρ , for some ρ in $(0, r]$, at any given pair (x_0, r) . On the other hand, there are plenty of simple examples in which any attempt at the converse fails. However, if one is interested in sparseness of a set at scale r , uniformly in x_0 (as we will be), the difference is much less pronounced.

We will formulate our regularity criterion as a no-blow-up criterion for a solution ω belonging to $C([0, T^*), L^\infty)$ where T^* is the first (possible) blow-up time. In this setting, it is convenient to introduce the concept of an ‘escape time’.

Definition 13. Let ω be in $C([0, T^*), L^\infty)$ where T^* is the first possible blow-up time. A time t in $(0, T^*)$ is an *escape time* if $\|\omega(\tau)\|_\infty > \|\omega(t)\|_\infty$ for any τ in (t, T^*) . (There are continuum-many escape times for solutions to (1).)

As in the previous section, we assume that the initial datum ω_0 is (in addition) in L^2 .

Then we can solve the 3D NSE locally-in-time, at any t in $(0, T^*)$, and uniqueness in conjunction with Theorem 9 will yield a lower bound on the radius of spatial analyticity near t . In particular, for an escape time t , we consider a temporal point $s = s(t)$ conforming to the requirement

$$(11) \quad s \text{ in } \left[t + \frac{1}{4C(M)\|\omega(t)\|_\infty}, t + \frac{1}{C(M)\|\omega(t)\|_\infty} \right];$$

then, a lower bound on the radius of spatial analyticity at s is given by

$$\frac{1}{2c(M)\|\omega(t)\|_\infty^{\frac{1}{2}}}.$$

Moreover, since t is an escape time, this lower bound can be replaced by

$$\frac{1}{2c(M)\|\omega(s)\|_\infty^{\frac{1}{2}}},$$

i.e., in this situation there is no need to worry about a time-lag—a lower bound on the analyticity radius at s is given in terms of the L^∞ -norm of the solution at s .

Henceforth, the norm of a vector (a, b, c) will be $\max\{|a|, |b|, |c|\}$.

The property of 1D sparseness will be assumed on the super-level sets of the positive and negative parts of the vorticity components ω_j at s (recall that for a scalar-valued function f , the positive and negative parts of f are given by $f^+(x) = \max(f(x), 0)$ and $f^-(x) = -\min(f(x), 0)$, respectively); more precisely, the sets of interests will be the sets $V_s^{j,\pm}$ defined by

$$(12) \quad V_s^{j,\pm} = \left\{ x \in \mathbb{R}^3 : \omega_j^\pm(x, s) > \frac{1}{2M}\|\omega(s)\|_\infty \right\}$$

(M is as in Theorem 9).

The last ingredient needed to set up our regularity criterion is an extremal property of the harmonic measure in the unit disk \mathbb{D} . Here, as in [Gr13], we are concerned with the harmonic measure in the complex plane \mathbb{C} ; for a suitable pair of sets (Ω, K) , the harmonic measure of K with

respect to Ω , evaluated at a point z , will be denoted by $h(z, \Omega, K)$. (For the basic properties of the harmonic measure in this setting, see, e.g., [Ahl, Ran].)

Proposition 14. [Sol] *Let λ be in $(0, 1)$, K a closed subset of $[-1, 1]$ such that $m^1(K) = 2\lambda$, and suppose that the origin is in $\mathbb{D} \setminus K$. Then*

$$h(0, \mathbb{D}, K) \geq h(0, \mathbb{D}, K_\lambda) = \frac{2}{\pi} \arcsin \frac{1 - (1 - \lambda)^2}{1 + (1 - \lambda)^2}$$

where $K_\lambda = [-1, -1 + \lambda] \cup [1 - \lambda, 1]$.

At this point, it is beneficial to set a specific set of constants: let M be the solution to the equation $\frac{1}{2}h^* + (1 - h^*)M = 1$ where $h^* = \frac{2}{\pi} \arcsin \frac{1 - (\frac{3}{4})^{\frac{2}{3}}}{1 + (\frac{3}{4})^{\frac{2}{3}}}$, and let $c(M)$ be as in Theorem 9 (note that $1 < M < \frac{3}{2}$).

Theorem 15. *Let ω be in $C([0, T^*), L^\infty)$ where T^* is the first possible blow-up time, and assume, in addition, that ω_0 is in L^2 (then, ω is automatically in $L^\infty((0, T^-), L^2)$ for any $0 < T^- < T^*$).*

Let t be an escape time, and suppose that there exists a temporal point $s = s(t)$ as in (11) such that for any spatial point x_0 , there exists a scale ρ , $0 < \rho \leq \frac{1}{2C(M)\|\omega(s)\|_\infty^{\frac{1}{2}}}$ and a direction \mathbf{d} with the property that the super-level set $V_s^{j,\pm}$ delineated in (12) is $1D(\frac{3}{4})^{\frac{1}{3}}$ -sparse around x_0 at scale ρ ; here, the pair (j, \pm) is chosen according to the following selection criterion: $|\omega(x_0, s)| = \omega_j^\pm(x_0, s)$.

Then T^ is not a blow-up time.*

Remark 16. Notice that—as far as the temporal intermittency goes—it is enough that the condition on local 1D sparseness holds at some $s(t)$ for a single escape time t (recall that there are continuum-many escape times; in particular, there are continuum-many escape times in any interval of the form $(T^* - \epsilon, T^*)$).

Proof. (The argument to be presented is a refinement of the argument introduced in [Gr13].)

Let t be an escape time and $s = s(t)$ satisfy the condition (11). The idea is to show that

$$(13) \quad \|\omega(s)\|_\infty \leq \|\omega(t)\|_\infty$$

utilizing the sparseness of the super-level set at s via the harmonic measure maximum principle; this would contradict the assumption that t is an escape and in turn the assumption that T^* is the first possible blow-up time.

Fix a spatial point x_0 and let \mathbf{d} be the direction of 1D sparseness around x_0 postulated in the theorem. Since the 3D NSE are translationally and rotationally invariant, and the coordinate transformations would change neither the lower bound on the analyticity radius nor the L^∞ -bound on the complexified solution (nor the sparseness), without loss of generality, assume that $x_0 = 0$ and

$\mathbf{d} = (1, 0, 0)$. (The argument is completely local, i.e., we are considering one spatial point x_0 at the time.)

Let us focus on the complexification of the real coordinate x_1 (regarding the other two variables as parameters), and view $\omega(s)$ as an analytic, \mathbb{C}^3 -valued function on the symmetric strip around the x_1 -axis of the width of at least

$$\frac{1}{c(M)\|\omega(s)\|_\infty^{\frac{1}{2}}}.$$

Since ω is analytic, the real and the imaginary parts of the component functions ω_j are harmonic, and their positive and negative parts subharmonic in the strip; in particular, the assumption on the scale of sparseness ρ implies that they are subharmonic in the open disc

$$D_\rho = \{z \in \mathbb{C} : |z| < \rho\}.$$

Select a pair (j, \pm) such that $|\omega(0, s)| = \omega_j^\pm(0, s)$. By the premise, the corresponding set $V_s^{j, \pm}$ is 1D $(\frac{3}{4})^{\frac{1}{3}}$ -sparse around 0 at scale ρ (in the direction e_1).

Define a set K by

$$K = \overline{V_s^{j, \pm} \cap (-\rho, \rho)}$$

(the complement is taken within the interval $[-\rho, \rho]$), and note that $m^1(K) \geq 2\rho(\frac{3}{4})^{\frac{1}{2}}$.

There are two disjoint scenarios to consider. In the first one, 0 is in K . Then,

$$|\omega(0, s)| = \omega_j^\pm(0, s) \leq \frac{1}{2M}\|\omega(s)\|_\infty \leq \frac{1}{2}\|\omega(t)\|_\infty;$$

i.e., the contribution of this particular x_0 is consistent with (13), and we can move on. In the second scenario, we can estimate the harmonic measure of K with respect to D_ρ at 0 utilizing the sparseness in conjunction with Proposition 14.

Recall that the harmonic measure is invariant with respect to any conformal map, and in particular with respect to the scaling map $\phi(z) = \frac{1}{\rho}z$; hence, Proposition 14 yields

$$h(0, D_\rho, K) \geq \frac{2}{\pi} \arcsin \frac{1 - (\frac{3}{4})^{\frac{2}{3}}}{1 + (\frac{3}{4})^{\frac{2}{3}}} = h^*.$$

The harmonic measure maximum principle applied to the subharmonic function ω_j^\pm then implies the following bound:

$$\omega_j^\pm(0, s) \leq h^* \frac{1}{2} \|\omega(t)\|_\infty + (1 - h^*) M \|\omega(t)\|_\infty = \left(\frac{1}{2} h^* + (1 - h^*) M \right) \|\omega(t)\|_\infty = \|\omega(t)\|_\infty;$$

the estimate on ω_j^\pm on the complement of K (within D_ρ) comes from the estimate on the complexified solution in Theorem 9, and the last equality from our choice of the parameters.

Consequently, the second scenario is also compatible with the inequality (13).

Since the point x_0 was an arbitrary point in \mathbb{R}^3 , this concludes the proof. □

4. SPATIAL COMPLEXITY OF THE VORTICITY COMPONENTS: AN A PRIORI BOUND

Concepts of a set being r -mixed (or ‘mixed to scale r ’) and r -semi-mixed appear in the study of mixing properties of incompressible (or nearly incompressible) flows, and in particular in the study of the optimal mixing strategies for passive scalars (e.g., density of a tracer) advected by an incompressible velocity field (see, e.g., [Br] and [IKX]).

Definition 17. Let $r > 0$. An open set S is r -semi-mixed if there exists δ in $(0, 1)$ such that

$$\frac{m^3(S \cap B(x, r))}{m^3(B(x, r))} \leq \delta$$

for every x in \mathbb{R}^3 . If the complement of S is r -semi-mixed as well, then S is said to be r -mixed.

Remark 18. Recalling the definitions from the previous section, note that if the set S is r -semi-mixed (with the ratio δ), then it is 3D δ -sparse around every point x_0 in \mathbb{R}^3 at scale r .

The following lemma is a vector-valued version of a positive scalar-valued lemma in [IKX] where the application was to the density of a tracer; a vectorial Besov space version was given in [FGL] in the context of the $B_{\infty, \infty}^{-1}$ -regularity criterion on the velocity. We present a detailed proof for completeness of the exposition.

Lemma 19. Let $r \in (0, 1]$ and f a bounded, continuous vector-valued function on \mathbb{R}^3 . Then, for any pair (λ, δ) , λ in $(0, 1)$ and δ in $(\frac{1}{1+\lambda}, 1)$, there exists a constant $c^*(\lambda, \delta) > 0$ such that if

$$\|f\|_{H^{-1}} \leq c^*(\lambda, \delta) r^{\frac{5}{2}} \|f\|_\infty$$

then each of the six super-level sets $S_\lambda^{i, \pm} = \{x \in \mathbb{R}^3 : f_i^\pm(x) > \lambda \|f\|_\infty\}$ is r -semi-mixed with the ratio δ .

Remark 20. Even in the scalar case, there could be a significant difference between semi-mixedness of the super-level sets of the absolute value of a sign-fluctuating function and semi-mixedness of the corresponding (cut at the same level) super-level sets of the positive and negative parts of the function. If we allow for discontinuities, the former may not be semi-mixed at all, while the latter may be r -semi-mixed at any predetermined scale. In the vectorial case, this phenomenon can easily manifest even within the class of smooth vector fields (comparing the full vectorial super-level sets with any of the six super-level sets of the positive and negative parts of the components). In particular, a duality argument of the type utilized in the lemma does not seem to lead to any quantifiable semi-mixedness of the full vectorial super-level sets.

Proof. Assume the opposite, i.e., there exists an index i such that either $S_\lambda^{i,+}$ or $S_\lambda^{i,-}$ is not r -semi-mixed with the ratio δ . Suppose that it is $S_\lambda^{i,+}$ (if it were $S_\lambda^{i,-}$, the only modification would be to replace the function ϕ below with $-\phi$).

Consequently, there exists a spatial point x_0 such that

$$m^3(S_\lambda^{i,+} \cap B(x_0, r)) > \delta V_3 r^3,$$

where V_3 denotes the volume of the unit ball in \mathbb{R}^3 .

Let ϕ be an H^1 -optimal, smooth radial (monotone) function, equal to 1 in $B(x_0, r)$, and vanishing outside $B(x_0, (1 + \eta)r)$ for some $\eta > 0$ (the value to be determined). Then, by duality,

$$(14) \quad \|f\|_{H^{-1}} \geq \frac{1}{\|\phi\|_{H^1}} \left| \int_{\mathbb{R}^3} f_i(x) \phi(x) dx \right|.$$

An explicit calculation of the H^1 -norm of ϕ yields

$$(15) \quad \|\phi\|_{H^1} \leq c(\eta) r^{\frac{1}{2}}$$

for a suitable $c(\eta) > 0$ (recall that r is in $(0, 1]$).

The objective is to obtain a lower bound on the numerator by performing a suitable domain-decomposition of the integral.

$$\left| \int_{\mathbb{R}^3} f_i(x) \phi(x) dx \right| \geq \int_{\mathbb{R}^3} f_i(x) \phi(x) dx \geq I - |II| - |III|,$$

where

$$I = \int_{S_\lambda^{i,+} \cap B(x_0, r)} f_i(x) \phi(x) dx,$$

$$II = \int_{B(x_0, r) \setminus S_\lambda^{i,+}} f_i(x) \phi(x) dx$$

and

$$III = \int_{(B(x_0, (1+\eta)r) \setminus B(x_0, r))} f_i(x) \phi(x) dx.$$

It is transparent that

$$\begin{aligned} I &= \int_{S_\lambda^{i,+} \cap B(x_0, r)} f_i(x) dx = \int_{S_\lambda^{i,+} \cap B(x_0, r)} f_i^+(x) dx \\ (16) \quad &> \lambda \|f\|_\infty m(S_\lambda^{i,+} \cap B(x_0, r)) \geq \lambda \delta V_3 r^3 \|f\|_\infty, \end{aligned}$$

$$\begin{aligned} |II| &= \left| \int_{B(x_0, r) \setminus S_\lambda^{i,+}} f_i(x) dx \right| \leq \|f\|_\infty \left(m(B(x_0, r)) - m(S_\lambda^{i,+} \cap B(x_0, r)) \right) \\ &\leq \|f\|_\infty (V_3 r^3 - \delta V_3 r^3) \\ (17) \quad &= (1 - \delta) V_3 r^3 \|f\|_\infty \end{aligned}$$

and

$$\begin{aligned} |III| &\leq \left| \int_{(B(x_0, (1+\eta)r) \setminus B(x_0, r))} f_i(x) dx \right| \\ &\leq \|f\|_\infty (m(B(x_0, (1+\eta)r)) - m(B(x_0, r))) \\ (18) \quad &\leq ((1+\eta)^3 - 1) V_3 r^3 \|f\|_\infty. \end{aligned}$$

Collecting the estimates (14), (15) and (16)–(18), it follows that

$$\|f\|_{H^{-1}} > c^*(\eta) r^{\frac{5}{2}} \|f\|_\infty (\lambda \delta + \delta - (1+\eta)^3).$$

Since $\delta > \frac{1}{1+\lambda}$ is postulated, the equation $(1+\eta)^3 = \frac{\delta(1+\lambda)+1}{2}$ has a unique solution $\eta = \eta(\lambda, \delta)$; this choice of η yields

$$(19) \quad \|f\|_{H^{-1}} > c^*(\lambda, \delta) r^{\frac{5}{2}} \|f\|_\infty$$

with $c^*(\lambda, \delta) = c^*(\eta) \frac{\delta(1+\lambda)-1}{2}$ (which is positive since $\delta > \frac{1}{1+\lambda}$). This produces a contradiction.

□

The next theorem is a simple consequence of the above lemma and a careful choice of parameters throughout the paper; it is designated a theorem because of its significance.

Theorem 21. *Let u be a Leray solution (a global-in-time weak solution satisfying the global energy inequality), and assume that ω is in $C((0, T^*), L^\infty)$ for some $T^* > 0$. Then for any τ in $(0, T^*)$ for which the scale r^* defined below is less or equal to one, the super-level sets*

$$V_\tau^{j,\pm} = \left\{ x \in \mathbb{R}^3 : \omega_j^\pm(x, \tau) > \frac{1}{2M} \|\omega(\tau)\|_\infty \right\}$$

are $3D \frac{3}{4}$ -sparse around any spatial point x_0 at scale

$$r^* = c(\|u_0\|_2) \frac{1}{\|\omega(\tau)\|_\infty^{\frac{2}{5}}}$$

where $c(\|u_0\|_2)$ is a constant depending only on the energy at time 0 (M is the parameter set preceding the statement of Theorem 15; recall that $1 < M < \frac{3}{2}$).

Proof. Notice that our choice of parameters implies that

$$\frac{3}{4} > \frac{1}{1 + \frac{1}{2M}},$$

and Lemma 19 is applicable.

Since we are on the whole space, an efficient way to estimate $\|\omega(\tau)\|_{H^{-1}}$ is by switching to the Fourier space where we have to estimate integrals of the form

$$I_{i,j} = \int \frac{1}{1 + |\xi|^2} |\widehat{\partial_i u_j}(\xi, \tau)|^2 d\xi.$$

This is plain since

$$I_{i,j} \leq \int \frac{|\xi_i|^2}{1 + |\xi|^2} |\widehat{u_j}(\xi, \tau)|^2 d\xi \leq \|\widehat{u(\tau)}\|_2^2 = \|u(\tau)\|_2^2 \leq \|u_0\|_2^2$$

(by the energy inequality).

Consequently, in order to satisfy all the assumptions in the lemma, it suffices to require

$$c \|u_0\|_2 \leq c^* \left(\frac{1}{1 + \frac{1}{2M}}, \frac{3}{4} \right) r^{\frac{5}{2}} \|\omega(\tau)\|_\infty,$$

which forces the choice of the scale of sparseness r^* in the theorem.

□

Remark 22. For our purposes, the restriction $r^* \leq 1$ in the theorem is irrelevant since we are only interested in temporal points τ leading to a possible blow-up time.

Remark 23. Some effort has been made to assure that both the cut-off levels for the super-level sets and the ratios of sparseness in Theorem 15 and Theorem 21 are identical, $\lambda = \frac{1}{2M}$ and $\delta = \frac{3}{4}$. Moreover, since the constants featured in either of the two scales of sparseness are either absolute (after our choice of parameters was made) constants or the constant depending only on the initial energy of the solution, the tolerance parameter c_0 depends only on the initial energy. Shortly, the regularity condition is given in the class $Z_{\frac{1}{2}}(\frac{1}{2M}, \frac{3}{4}; c_0(\|u_0\|_2))$, and the *a priori* bound in the class $Z_{\frac{2}{5}}(\frac{1}{2M}, \frac{3}{4}; c_0(\|u_0\|_2))$.

5. EPILOGUE

The authors’ goal was to present a mathematical framework—the scale of classes Z_α —in which one could break the archetypal 3D NS scaling barrier in the context of a blow-up-type argument. More precisely—in this ambiance—we showed that the *a priori* bound can be shifted by an algebraic factor, from $Z_{\frac{1}{3}}$, which corresponds to the classical *a priori* bounds, to $Z_{\frac{2}{5}}$; recall that the regularity class in this setting is $Z_{\frac{1}{2}}$, which corresponds to the classical regularity criteria, and can be viewed as a statement that the radius of spatial analyticity is a *bona fide* diffusion scale.

Parallel to the continued mathematical efforts, there is an effort directed at gaining some insight into a ‘typical value’ of α by performing high-resolution computational simulations of turbulent flows, and trying to identify the scaling properties of the scale of sparseness of the super-level sets of the positive and negative parts of the vorticity components with respect to the diffusion scale $\frac{\nu^{\frac{1}{2}}}{\|\omega\|_\infty^{\frac{1}{2}}}$. This is currently underway as a joint project with the complexity group of J. Mathiesen at the Niels Bohr Institute (NBI) at Copenhagen, and in particular, with M. Mizstal, and the ScienceAtHome group at Aarhus University led by J. Sherson; J. Rafner, of both the NBI and Aarhus is the project coordinator.

Acknowledgments

The work of A.F. is supported in part by the National Science Foundation grant DMS-1418911. The work of Z.G. is supported in part by the National Science Foundation grant DMS - 1515805 and the Lundbeck Foundation grant R217-2016-446 (this is the collaborative grant with the NBI and Aarhus). Z.G. thanks the Program in Applied and Computational Mathematics at Princeton for their hospitality in Spring 2017 when the paper was finalized.

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